Lesson 1: Construct an Equilateral Triangle

Classwork

Opening Exercise

Joe and Marty are in the park playing catch. Tony joins them, and the boys want to stand so that the distance between any two of them is the same. Where do they stand?

How do they figure this out precisely? What tool or tools could they use?

Fill in the blanks below as each term is discussed:

1. ______________________ The _______ between points A and B is the set consisting of A, B, and all points on the line AB between A and B.

2. ______________________ A segment from the center of a circle to a point on the circle.

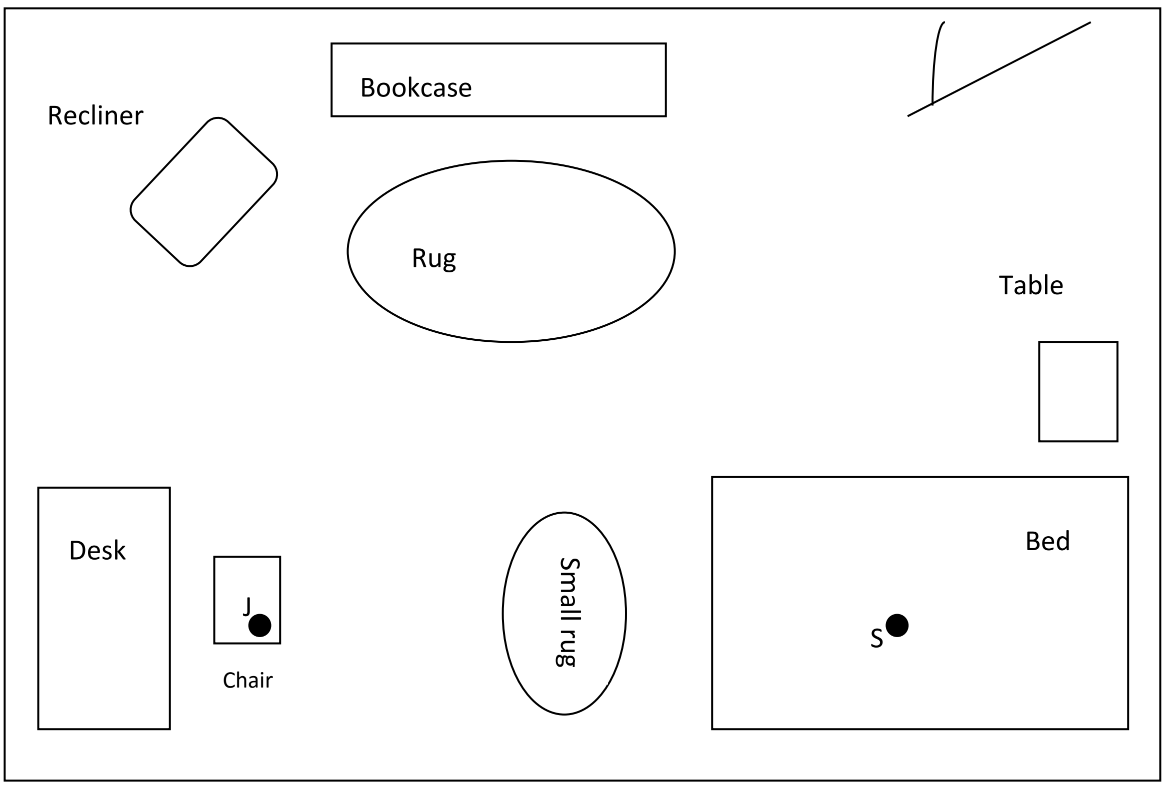
3. ______________________ Given a point C in the plane and a number $r > 0$, the _______ with center C and radius r is the set of all points in the plane that are distance r from point C.

Note that because a circle is defined in terms of a distance, r, we will often use a distance when naming the radius (e.g., "radius AB"). However, we may also refer to the specific segment, as in "radius $\overline{AB}$."

Example 1: Sitting Cats

You will need a compass and a straightedge.

Margie has three cats. She has heard that cats in a room position themselves at equal distances from one another and wants to test that theory. Margie notices that Simon, her tabby cat, is in the center of her bed (at S), while JoJo, her Siamese, is lying on her desk chair (at J). If the theory is true, where will she find Mack, her calico cat? Use the scale drawing of Margie's room shown below, together with (only) a compass and straightedge. Place an M where Mack will be if the theory is true.



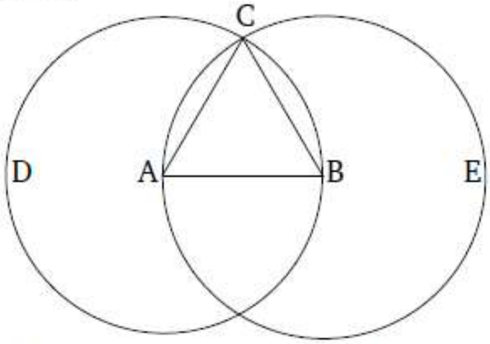
Mathematical Modeling Exercise: Euclid, Proposition 1

Let's see how Euclid approached this problem. Look at his first proposition, and compare his steps with yours.

Proposition 1

In this margin, compare your steps with Euclid's.

To construct an equilateral triangle on a given finite straight-line.



Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line AB.

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another,† to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

Geometry Assumptions

In geometry, as in most fields, there are specific facts and definitions that we assume to be true. In any logical system, it helps to identify these assumptions as early as possible since the correctness of any proof hinges upon the truth of our assumptions. For example, in Proposition 1, when Euclid says, "Let AB be the given finite straight line," he assumed that, *given any two distinct points, there is exactly one line that contains them*. Of course, that assumes we have two points! It is best if we assume there are points in the plane as well: *Every plane contains at least three non-collinear points.*

Euclid continued on to show that the measures of each of the three sides of his triangle are equal. It makes sense to discuss the measure of a segment in terms of distance. *To every pair of points A and B, there corresponds a real number* $\text{dist}(A,B) \geq 0$, *called the distance from A to B.* Since the distance from A to B is equal to the distance from B to A, we can interchange A and B: $\text{dist}(A,B) = \text{dist}(B,A)$. Also, *$A$ and B coincide if and only if* $\text{dist}(A,B) = 0$.

Using distance, we can also assume that *every line has a coordinate system*, which just means that we can think of any line in the plane as a number line. Here's how: Given a line, l, pick a point A on l to be "0," and find the two points B and C such that $\text{dist}(A,B) = \text{dist}(A,C) = 1$. Label one of these points to be 1 (say point B), which means the other point C corresponds to -1. Every other point on the line then corresponds to a real number determined by the (positive or negative) distance between 0 and the point. In particular, if after placing a coordinate system on a line, if a point R corresponds to the number r, and a point S corresponds to the number s, then the distance from R to S is $\text{dist}(R,S) = |r - s|$.

History of Geometry: Examine the site http://geomhistory.com/home.html to see how geometry developed over time.

Relevant Vocabulary

Geometric Construction: A *geometric construction* is a set of instructions for drawing points, lines, circles, and figures in the plane.

The two most basic types of instructions are the following:

1. Given any two points A and B, a ruler can be used to draw the line AB or segment $\overline{AB}$.
2. Given any two points C and B, use a compass to draw the circle that has its center at C that passes through B. (Abbreviation: Draw circle C: center C, radius CB.)

Constructions also include steps in which the points where lines or circles intersect are selected and labeled. (Abbreviation: Mark the point of intersection of the lines AB and PQ by X, etc.)

Figure: A (two-dimensional) *figure* is a set of points in a plane.

Usually the term figure refers to certain common shapes such as triangle, square, rectangle, etc. However, the definition is broad enough to include any set of points, so a triangle with a line segment sticking out of it is also a figure.

Equilateral Triangle: An *equilateral triangle* is a triangle with all sides of equal length.

Collinear: Three or more points are collinear if there is a line containing all of the points; otherwise, the points are non-collinear.

Length of a Segment: The *length of the segment* $\overline{AB}$ is the distance from A to B and is denoted AB. Thus, $AB = \text{dist}(A, B)$.

In this course, you will have to write about distances between points and lengths of segments in many, if not most, Problem Sets. Instead of writing $\text{dist}(A, B)$ all of the time, which is a rather long and awkward notation, we will instead use the much simpler notation AB for both distance and length of segments. Even though the notation will always make the meaning of each statement clear, it is worthwhile to consider the context of the statement to ensure correct usage.

Here are some examples:

- $\overleftrightarrow{AB}$ intersects… $\overleftrightarrow{AB}$ refers to a line.
- $AB + BC = AC$ Only numbers can be added and AB is a length or distance.
- Find $\overline{AB}$ so that $\overline{AB} \parallel \overline{CD}$. Only figures can be parallel and $\overline{AB}$ is a segment.
- $AB = 6$ AB refers to the length of the segment AB or the distance from A to B.

Here are the standard notations for segments, lines, rays, distances, and lengths:

- A ray with vertex A that contains the point B: $\overrightarrow{AB}$ or "ray AB"
- A line that contains points A and B: $\overleftrightarrow{AB}$ or "line AB"
- A segment with endpoints A and B: $\overline{AB}$ or "segment AB"
- The length of segment $\overline{AB}$: AB
- The distance from A to B: $\text{dist}(A, B)$ or AB

Coordinate System on a Line: Given a line l, a *coordinate system on* l is a correspondence between the points on the line and the real numbers such that: (i) to every point on l, there corresponds exactly one real number; (ii) to every real number, there corresponds exactly one point of l; (iii) the distance between two distinct points on l is equal to the absolute value of the difference of the corresponding numbers.

Problem Set

1. Write a clear set of steps for the construction of an equilateral triangle. Use Euclid's Proposition 1 as a guide.

2. Suppose two circles are constructed using the following instructions:

   Draw circle: Center A, radius AB.

   Draw circle: Center C, radius CD.

   Under what conditions (in terms of distances AB, CD, AC) do the circles have

   a. One point in common?
   b. No points in common?
   c. Two points in common?
   d. More than two points in common? Why?

3. *You will need* a compass and straightedge.

   Cedar City boasts two city parks and is in the process of designing a third. The planning committee would like all three parks to be equidistant from one another to better serve the community. A sketch of the city appears below, with the centers of the existing parks labeled as P_1 and P_2. Identify two possible locations for the third park, and label them as P_{3a} and P_{3b} on the map. Clearly and precisely list the mathematical steps used to determine each of the two potential locations.